Conformal Defects in Nonlocal $O(N)$-invariant theories

ICFP M1 Internship

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Conformal Defects

Why study defects?

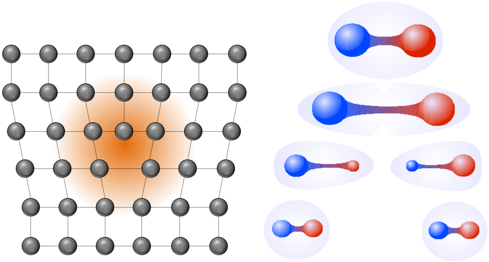

Figure: Left: dislocation in crystal lattice. Right: color confinement in QCD.

Conformal Defects

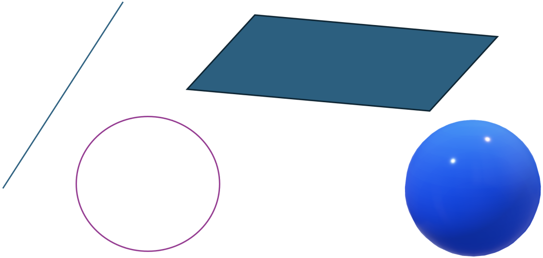

Figure: Types of one-dimensional and two-dimensional defects.

- Defect $\rightarrow$ p-dimensional subspace of d-dimensional bulk space.
- Broken symmetry: $SO(d+2) \rightarrow SO(d-p) \times SO(p+2)$

Correlators and DCFT data

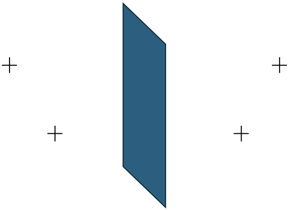

Figure: Distances between operator insertions and defect add length scales for correlators.

$$\langle \mathcal{O}_1(x) \rangle = \frac{a_{\mathcal{O}_1}}{|x|_{d-p}^{\Delta_{\mathcal{O}_1}}} \tag{1}$$

Conformal Defects

Existence of non-trivial defects (Lauria et al. 2021)

In integer dimensions less than four, there are no non-trivial defects in a **local free** theory (slightly weaker statement than in the paper, but this is enough for our purposes).

2 loopholes: **nonlocal** or **interacting** theory.

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The Localized Magnetic Field

Long-range $O(N)$ model

Generalized free field + ϕ^4 interactions:

$$S = \frac{\mathcal{N}_\sigma}{2} \int d^d x d^d y \frac{\phi_a(x)\phi_a(y)}{|x-y|^{d+\sigma}} + \frac{\lambda_0}{4!} \int d^d x \left(\phi_a(x)^2\right)^2 \tag{2}$$

can also be written

$$S = \frac{1}{2} \int d^d x \phi_a(x) \mathcal{L}_\sigma \phi_a(x) + \frac{\lambda_0}{4!} \int d^d x \left(\phi_a(x)^2\right)^2 \tag{3}$$

where $\mathcal{L}_\sigma \phi(x) = \frac{2^\sigma \Gamma\left(\frac{d+\sigma}{2}\right)}{\pi^{\frac{d}{2}} \Gamma\left(-\frac{\sigma}{2}\right)} \int d^d y \frac{\phi(y)}{|x-y|^{d+\sigma}}$

The Localized Magnetic Field

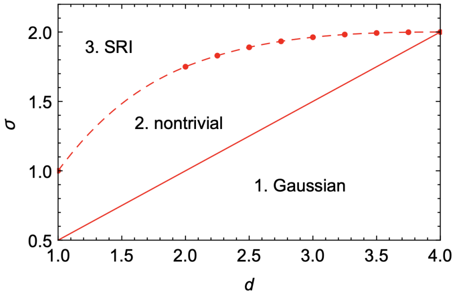


Figure: Phases of the critical long-range Ising model.

LRI fixed point shown to be conformally invariant (Paulos et al. 2016).

The Localized Magnetic Field

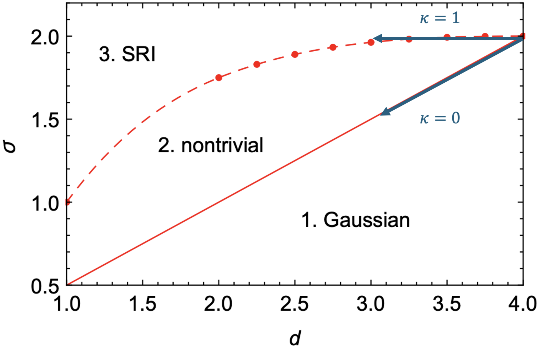
- Extend results on localized magnetic field (Allais and Sachdev 2014; Cuomo, Komargodski, and Mezei 2022) to nonlocal case.

$$S = \frac{1}{2}\int d^d x \phi_a(x)\mathcal{L}_\sigma \phi_a(x) + \frac{\lambda_0}{4!}\int d^d x \left(\phi_a(x)^2\right)^2 + h_0 \int dx \phi_1(x) \quad (2)$$

- Requiring classical marginality of bulk and defect interactions forces us to look at $d \to 4$, $\sigma \to d/2$.

The Localized Magnetic Field

ε-**expansion**: $d = 4 - \varepsilon$, $\sigma = (d + \kappa\varepsilon)/2$



The Localized Magnetic Field

ε-**expansion**: $d = 4 - \varepsilon$, $\sigma = (d + \kappa\varepsilon)/2$

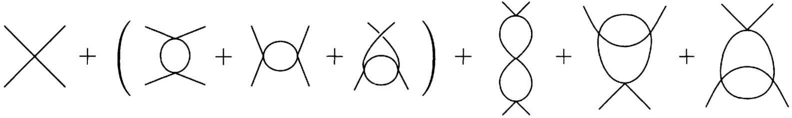


Figure: Four-point function of $\phi_a(x)$ at 2-loop order (Peskin and Schroeder 1995).

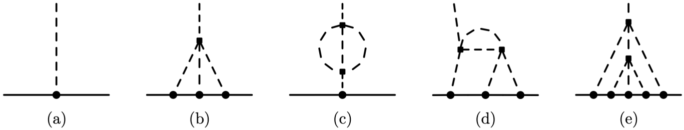


Figure: One-point function of $\phi_1(x)$ at 2-loop order (Cuomo, Komargodski, and Mezei 2022).

The Localized Magnetic Field

ε-**expansion**: $d = 4 - \varepsilon$, $\sigma = (d + \kappa\varepsilon)/2$

$$\beta_\lambda = -\kappa\varepsilon\lambda + \frac{N+8}{3}\frac{\lambda^2}{(4\pi)^2} - \frac{10N + 44 + \kappa(26N+124)}{9(1+3\kappa)}\frac{\lambda^3}{(4\pi)^4} + \mathcal{O}\left(\frac{\lambda^4}{(4\pi)^6}\right) \tag{3}$$

$$\begin{aligned}\beta_h = &-\frac{\varepsilon\,(1+\kappa)}{4}h + \frac{\lambda}{(4\pi)^2}\frac{h^3}{6} + \frac{\lambda^2}{(4\pi)^4}\left(\frac{(N+2)\,\kappa h}{9\,(1+3\kappa)}\right.\\ &\left.-\frac{(N+8)\left(1+3\kappa^2-(1-\kappa)\left(\gamma-\log\left(4\pi\right)\right)\right)}{36\kappa\,(1+3\kappa)}h^3 - \frac{h^5}{12}\right)\\ &+\mathcal{O}\left(\frac{\lambda^3}{(4\pi)^6}\right)\end{aligned} \tag{4}$$

ε-**expansion**: $d = 4 - \varepsilon$, $\sigma = (d + \kappa\varepsilon)/2$

$$\frac{\lambda_*}{(4\pi)^2} = \frac{3}{N+8}\kappa\varepsilon + \frac{6}{(N+8)^3}\frac{5N + 22 + \kappa\left(13N + 62\right)}{1+3\kappa}\kappa^2\varepsilon^2 + \mathcal{O}\left(\varepsilon^3\right) \quad (3)$$

$$h_*^2 = \frac{(N+8)(1+\kappa)}{2\kappa} + \frac{\left(5 + 2(1-\kappa^2)(\log(4\pi) - \gamma)\right)(N+8)}{8\kappa(1+3\kappa)}\varepsilon$$
$$+ \frac{(15\kappa^2 + 27\kappa + 17)N^2 + 8(15\kappa^2 + 36\kappa + 29)N + 48(9\kappa^2 + 22\kappa + 19)}{8(1+3\kappa)(N+8)}\varepsilon$$
$$+ \mathcal{O}(\varepsilon^2) \quad (4)$$

The Localized Magnetic Field

DCFT Data

$$\langle[\phi_a(x)]\rangle = \delta_{a,1}\frac{\sqrt{N_\phi}a_\phi}{|x|_{d-p}^{\Delta_\phi+\gamma_\phi}} \tag{5}$$

Hence:

$$\begin{aligned}
a_\phi^2 &= \frac{(\kappa+1)(N+8)}{8\kappa} \\
&+ \left[\frac{3\kappa^2\left(N^2(3+\log(4))+8N(1+\log(16))+16(1+\log(256))\right)}{32(3\kappa+1)(N+8)}\right. \\
&+\frac{\kappa\left(9N^2+14(N+8)^2\log(2)-96\right)}{32(3\kappa+1)(N+8)} \\
&\left.+\frac{10(N+8)^2\log(2)-N(N+56)-240}{32(3\kappa+1)(N+8)}+\frac{(N+8)(\log(4)-1)}{32\kappa(3\kappa+1)}\right]\varepsilon+\mathcal{O}\left(\varepsilon^2\right)
\end{aligned} \tag{6}$$

DCFT Data

Verify that the g-theorem holds perturbatively (Cuomo, Komargodski, and Raviv-Moshe 2022).

Figure: Feynman diagrams in the g-function up to one-loop order.

Using the relations between bare and renormalized couplings, we arrive at

$$\log g = -\frac{(1+\kappa)\varepsilon}{16}h^2 + \frac{\kappa h^4 \lambda_*}{192\pi^2(3\kappa+1)} + \mathcal{O}\left(\lambda_*^2\right) \tag{5}$$

DCFT Data

Using the value of the defect fixed point coupling h_* previously derived:

$$\log g_{\mathrm{IR}} = -\frac{(\kappa+1)^3(N+8)}{32\kappa(3\kappa+1)}\varepsilon + \mathcal{O}\left(\varepsilon^2\right) < 0 = \log g_{\mathrm{UV}} \tag{5}$$

Hence the g-theorem holds true perturbatively.

The Localized Magnetic Field

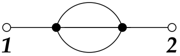
Wavefunction renormalization?

Case of nonlocal theories

In a field theory admitting a nonlocal Lagrangian describing a field ϕ, ϕ does not renormalize.

Bulk long-range $O(N)$ model at fixed finite d:

$$\frac{N+2}{3}\frac{1}{6}$$



$$= \lambda^2 \frac{(N+2)\Gamma\left(-\frac{d}{4}\right)}{2^d 18 (2\pi)^{\frac{3d}{2}} \Gamma\left(\frac{3d}{4}\right)} \frac{1}{|x|^{\frac{d}{2}}} + \mathcal{O}(\varepsilon) \tag{6}$$

Wavefunction renormalization?

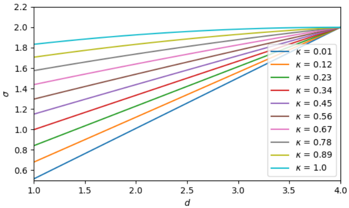
Perturbation of local theory

$$\mathcal{L}_{\mathrm{GFF}} \propto \int d^d p \phi(p) |p|^{2-\frac{1-\kappa}{2}\varepsilon} \phi(-p)$$
$$= \int d^d p \phi(p) |p|^2 \phi(-p) - \frac{1-\kappa}{2}\varepsilon \int d^d p \phi(p) |p|^2 \ln |p| \phi(-p) + \mathcal{O}\left(\varepsilon^2\right) \quad (6)$$

Wavefunction renormalization?

Some trajectories can be considered long-range.

$$\kappa_1 = \kappa - \frac{4\kappa^3 (N+2)}{(1+3\kappa)(N+8)^2}\varepsilon + \mathcal{O}\left(\varepsilon^2\right) \tag{6}$$



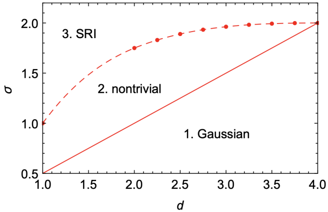


Figure: Left: The trajectories $\sigma(d)$ such that in the ε-expansion, $\gamma_\phi = 0$ in the case $N = 1$. Right: Phases of the critical long-range Ising model.

Expanding around DCFT solution

$$\phi_1(x) = \frac{a_\phi \sqrt{N_\phi}}{|x|_{d-1}^{\Delta_\phi}} + \delta\phi_1 \tag{7}$$

Next, we use the fact that classically $\langle \phi_1^3 \rangle = \langle \phi_1 \rangle^3$ and use equation of motion:

$$\mathcal{L}_\sigma \langle \phi_1 \rangle = -\frac{\lambda_0}{3!} \langle \phi_1^3 \rangle \tag{8}$$

We can then solve for a_ϕ by plugging (7) into the equation of motion:

$$a_\phi^2 = -\frac{6\mathcal{N}_\sigma}{\lambda_0 N_\phi} \frac{\sqrt{\pi}\,\Gamma\left(\frac{d+\sigma-1}{2}\right)}{\Gamma\left(\frac{d+\sigma}{2}\right)} \frac{w^{(d-1)}_{d+\sigma-1} w^{(d-1)}_{\frac{d-\sigma}{2}}}{w^{(d-1)}_{\frac{d+\sigma}{2}}} = \frac{(1+\kappa)(N+8)}{8\kappa} + \mathcal{O}(\varepsilon) \tag{9}$$

The Localized Magnetic Field

Expanding around DCFT solution

$$S = \frac{1}{2}\int d^d x \delta\phi_a(x)\mathcal{L}_\sigma \delta\phi_a(x) + \frac{\lambda_0}{4!}\int d^d x \left(\frac{2N_\phi a_\phi^2 \left(\delta\phi_a(x)^2 + 2\delta\phi_1(x)^2\right)}{|x|_{d-1}^{2\Delta_\phi}} + \frac{4\sqrt{N_\phi} a_\phi \delta\phi_1(x)^3}{|x|_{d-1}^{\Delta_\phi}} + \left(\phi_a(x)^2\right)^2 \right) \tag{7}$$

- h absent from this new theory
- Constraining solution to equation of motion via defect ansatz + bulk fixed point → defect fixed point.
- Behavior close to $d = 3$?

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Interaction with a GFF on a Defect

Setup

Add a second field on the defect, impose marginality and unitarity:

$$S = \int_d \left[\frac{1}{2}\phi\mathcal{L}_{d-2\Delta_\phi}\phi + \frac{\lambda}{4!}\phi^4\right] + \frac{1}{2}\int_p \psi\mathcal{L}_{p-2\Delta_\psi}\psi + \frac{g}{2}\int_p \phi^a\psi^b \tag{8}$$

$$\Delta_\psi = \frac{p - a\Delta_\phi}{b} = \frac{4p - ad}{4b}, \quad \begin{cases} ad \leq 4p \\ ad \leq 4p - 2b(p-2) \end{cases} \tag{9}$$

	$a = 1$	$a = 2$
$p = 1, d = 3$ or 4	Any b	
$p = 2, d = 3$ or 4	Any b	Any b
$p = 3, d = 4$	$b \leq 4$	$b \leq 2$

Figure: Table of parameters which yield unitary theories

Interaction with a GFF on a Defect

Matchup with localized magnetic field?

E.g. for $(a, b) = (1, 2)$:

$$S_{\text{eff}}[\phi] = \frac{1}{2} \int d^d x \phi \mathcal{L}_\sigma \phi + \frac{1}{2} \ln \left(1 + g_0 \int d^p x \phi(x) \right) \tag{10}$$

g renormalizes and one can show that the fixed point is infinitesimal.

$$S_{\text{eff}} = \frac{1}{2} \int d^d x \phi \mathcal{L}_\sigma \phi + \frac{g_0}{2} \int d^p x \phi(x) + \dots \tag{11}$$

Matchup at lowest order only.

Interaction with a GFF on a Defect

Matchup with localized magnetic field?

g-function readily accessible from the effective theory:

$$g = \sum_{k=0}^{\infty} \frac{(4k)!}{(2k)!k!} \left[g_0^2 2^{-3-2\Delta_\phi} R^{2-2\Delta_\phi} \pi \frac{\Gamma\left(\frac{1}{2} - \Delta_\phi\right)\sqrt{\pi}}{\Gamma\left(1 - \Delta_\phi\right)} \right]^k \tag{10}$$

The combinatorics prevent an exact matchup. Compare with result from localized magnetic field in free bulk:

$$g = \exp\left[\pi h^2 2^{1-2\Delta_\phi} R^{2-2\Delta_\phi} N_\phi \frac{\Gamma\left(\frac{1}{2} - \Delta_\phi\right)\sqrt{\pi}}{\Gamma\left(1 - \Delta_\phi\right)} \right] \tag{11}$$

Interaction with a GFF on a Defect

A trick for renormalization

Consider $O(x) = \phi^{a-1}(x)\psi^b(x)$. Equation of motion:

$$\mathcal{L}_{d-2\Delta_\phi}\phi(x) = -\frac{g_0}{2}aO(x)\delta^{(d-p)}(x) \tag{12}$$

Nonlocal theories do not renormalize $\rightarrow Z_g Z_O = 1$. Reasonable ansatz for Z_O:

$$Z_O = 1 - \frac{\alpha g^n}{\varepsilon} + \mathcal{O}\left(g^{n+1}\right) \tag{13}$$

$$\beta_g = -\varepsilon g + \alpha n g^{n+1} + \mathcal{O}\left(g^{n+1}\right) \tag{14}$$

and such a β-function leads to a non-trivial fixed point $g_*^n = \varepsilon/(\alpha n) + \mathcal{O}\left(\varepsilon^2\right)$.

Interaction with a GFF on a Defect

Trick to derive observables

- Invert the previous equation of motion at the fixed point for x in the bulk:

$$\phi(x) = -a\mathcal{N}_{2\Delta_\phi - d}\frac{g_0}{2}\int d^p y \frac{O(y)}{\left(|x|^2_{d-p} + |y|^2_p\right)^{\Delta_\phi}} \tag{15}$$

- Allows to easily deduce the one-point function of ϕ^2.

$$\langle \phi^2(x) \rangle = g_0^2 \frac{a^2 \mathcal{N}^2_{2\Delta_\phi - d}}{4} \frac{\pi^p \Gamma\left(\frac{p}{2}\right)\Gamma\left(\Delta_\phi - \frac{p}{2}\right)}{\Gamma\left(\Delta_\phi\right)(p-1)!} \frac{N_O}{|x|^{2\Delta_\phi}_{d-p}} = \frac{a_{\phi^2} N_\phi}{|x|^{2\Delta_\phi}_{d-p}} \tag{16}$$

Interaction with a GFF on a Defect

Trick to derive observables

- Non-trivial test of conformality:

$$\begin{aligned} \langle \phi(x) O(y) \rangle_D &= -a\mathcal{N}_{2\Delta_\phi - d} \frac{g_0}{2} \int \frac{d^p z}{|x-z|^{2\Delta_\phi} |z-y|^{2(p-\Delta_\phi)}} \\ &\propto \delta^{(p)}(x-y) = 0 \end{aligned} \tag{15}$$

- Few simple cases where renormalization of O is straightforward $(a, b) = (1, 2), (2, 1), (2, 2)$.

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Conclusion

- Nonlocality provides interesting dynamics + existence of non-trivial defects in free theory.
- Results from standard defects in local theories seem generalizable to nonlocal case. Could also look at boundary with an interaction going like ϕ^3 in $d = 4 - \varepsilon$[1].
- New family of defects with interesting dynamics close to arbitrary $3 \leq d \leq 4$, without reducing to the localized magnetic field or quadratic surface defect as $d \to 4$.
- Further work → large N expansion[2], analytic bootstrap[3].

[1] Harribey, Klebanov, and Sun 2023.
[2] Moshe and Zinn-Justin 2003.
[3] Bianchi, Bonomi, and Sabbata 2023.

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